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On Intrinsic Randomness of Dynamical Systems

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We discuss the problem of nonunitary equivalence, via positivity-preserving similarity transformations, between the unitary groups associated with deterministic dynamical evolution and semigroups associated with stochastic processes. Dynamical systems admitting such nonunitary equivalence with stochastic Markov processes are said to be *intrinsically random*. In a previous work, it was found that the so-called Bernoulli systems (discrete time) are intrinsically random in this sense. This result is extended here by showing that a more general class of dynamical systems—the so-called K systems and K flows—are intrinsically random. The connection of intrinsic randomness with local instability of motion is briefly discussed. We also show that Markov processes associated through nonunitary equivalence to *nonisomorphic* K flows are necessarily nonisomorphic.

KEY WORDS: Dynamical systems; Markov processes; K flows; H theorem; time operator; irreversibility; instability.

1. INTRODUCTION

The study of the possible connections that may exist between deterministic dynamics and probabilistic processes is of obvious importance for the foundation of nonequilibrium statistical mechanics. As is well known, stochastic Markov processes provide the best possible models to represent irreversible evolution admitting a Lyapounov functional or $\mathcal{H}$ function. The important question, thus, is how is the passage from deterministic dynamics to probabilistic Markov processes to be achieved?

It is generally believed that probabilistic processes can come from

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deterministic dynamics only as a result of some form of "coarse graining" and approximations which necessarily involve a loss of information. In this connection let us recall that the generally adopted procedure for obtaining, say, a Master equation from dynamics involves an initial "contraction of description" brought about by a projection operator followed by suitable limit procedure designed to restore the Markovian character of the evolution which is generally lost in the process of "contraction of description."

Recently a totally different possibility for obtaining stochastic Markovian evolution from deterministic dynamics has been proposed (see Ref. 1 and the earlier discussion in Ref. 2). The basic idea of this approach is to obtain a stochastic Markov process from a deterministic dynamics simply through a "change of representation" brought about by a (necessarily) nonunitary but *invertible* similarity transformation. More explicitly, let U_t denote the unitary group (acting on the space of distribution functions on the phase space) which is induced from the dynamical group T_t describing the (deterministic) motion of the phase points. One considers now the possibility of defining an *invertible* similarity transformation Λ mapping states (i.e., positive and normalized distribution functions) to states such that $\Lambda U_t \Lambda^{-1} \equiv W_t^*$ represents the semigroups of a stochastic Markov process. The existence of such a Λ means that under the "change of representation" $\rho \to \Lambda\rho$ the originally given deterministic dynamics described by the unitary group U_t is transformed into the stochastic Markovian dynamics associated with W_t^*. The important point to note is that this scheme for the passage from deterministic to stochastic dynamics involves no approximation and (as the required *invertibility* of Λ assures) no "contraction of description" or coarse graining.

For this reason dynamical systems admitting the existence of a similarity transformation Λ with the above-stated properties have been termed *intrinsically random dynamical systems* in Ref. 1.

How can it be possible that a "change of representation" can lead from deterministic to stochastic evolution? Should not a change of representation in classical mechanics be given simply by a canonical transformation of the phase space into itself? As suggested in Ref. 1 the answer to this is to be found in the concept of instability of motion. If the dynamical motion is sufficiently unstable so that each open region of the phase space, *no matter how small*, rapidly spreads out in time to far-separated regions in phase space then, obviously, one cannot arrive at the concept of the phase space trajectories by considering the motion of smaller and smaller initial regions of the phase space. Now, a physical measurement is necessarily of finite accuracy and hence can determine the initial conditions of the system to lie in only an open region of the phase space, which may be taken to be arbitrarily small but which can never be reduced to an individual phase point. This limitation on the possible physical measurements is of no consequence for stable dynamical evolution as one can arrive at the concept of phase space trajectory as the idealized limit of the motion of smaller and smaller phase space regions. But as we have seen, this is no longer possible in the case of sufficiently unstable dynamical motion. Thus for such systems the very concept of deterministic motion along phase space trajectories becomes an idealization which is beyond the possibility of physical realization *even as a limiting case*. As a consequence the *fundamental* objects of dynamical theory must not, now, be the individual phase points or trajectories but open regions or more generally, the distribution functions on the phase space: A change of representation of dynamics may therefore correspond now to more general transformations acting on the distribution function than those given by the canonical transformations of the phase space into itself.

According to the viewpoint expressed above the appearance of strong forms of instability of motion marks the breakdown of the deterministic description of dynamics given in terms of the phase space trajectories. In this situation it is both possible and natural to pass to a new representation of dynamics (provided by necessarily "nonlocal" transformations on the phase space) which eliminates from the theory the physically unrealizable idealization of determiministic motion along phase space trajectories and results in a stochastic Markov process. The notion of intrinsic randomness formalized this idea.

The consistency of this conception of *the close link between* intrinsic randomness and instability is supported by the result that the instability leading to mixing on phase space (in the sense of ergodic theory) is a *necessary* condition for the system to be intrinsically random.[1] As regards sufficiency, it was shown in Ref. 1 that the so-called Bernoulli systems (discrete time) are intrinsically random. (Actually only the simplest system of the Bernoulli systems—the so-called baker's transformation—is explicitly considered in Ref. 1. But as emphasized there it is not difficult to see that the same considerations can be generalized to an arbitrary Bernoulli system.[16]) The question as to whether Bernoulli flow (continuous time) and more generally the K systems and K flows are also intrinsically random, however, remained unsettled. The main purpose of this paper is to answer this question in the affirmative.

Let us mention that a great variety of systems of physical interest—infinite ideal gas and hard rods system,[3,4] motion of a billiard ball on a table with convex obstacle,[5,6] a hard sphere gas,[7] and geodesic flow on a manifold of negative curvature,[8,9] etc.—are known to be K flows and even Bernoulli. In view of the intrinsic randomness of K flows, one can associate with these systems stochastic Markov processes that are obtained from the *underlying* deterministic motion through nonunitary but invertible similarity transformations. It would be interesting to examine the physical meaning of these Markov processes and discuss questions such as the conditions under which they are diffusion processes.

2. INTRINSIC RANDOMNESS, OPERATOR TIME, AND INSTABILITY OF MOTION

In this section we recall the precise definition of intrinsic randomness and the "canonical" procedure for constructing the transformation Λ which leads from deterministic to stochastic dynamics. For more details see Refs. 1 and 10.

Let us consider an abstract dynamical system $\{\Gamma, \mathfrak{B}, \mu, T_t\}$. Here Γ denotes the phase space of the system equipped with the σ-algebra $\mathfrak{B}$ of subsets of Γ, T_t a group of measurable motions mapping Γ onto itself and preserving the measure μ. For example, Γ could be the energy surface of a classical dynamical system, T_t the group of dynamical evolution, and μ the invariant measure whose existence is assured by Liouville's theorem. For convenience, we shall assume the measure μ to be normalized: $\mu(\Gamma) = 1$. As is well known, the evolution of density functions ρ in L^2_μ under the given deterministic dynamics is described by the unitary group U_t induced by T_t:

$$(U_t\rho)(\omega) = \rho(T_{-t}\omega), \qquad \omega \in \Gamma$$

Every deterministic evolution thus defines a unitary group. Conversely (under the mild assumption that $(\Gamma, \mathfrak{B}, \mu)$ is a standard measure space) every unitary group which preserves positivity (i.e., maps nonnegative functions to nonnegative functions and leaves the constant functions unchanged) is induced by a group T_t of measure-preserving transformation on Γ (see, for example, Ref. 11).

On the other hand, stochastic Markov processes on the state space Γ, preserving μ, are associated with contraction semigroups of L^2_μ.[12] In fact, let $P(t, \omega, \Delta)$ denote the probability of transition from the point $\omega \in \Gamma$ to the region $\Delta \in \mathfrak{B}$ in time t. Then the operators W_t defined by

$$(W_t f)(\omega) = \int f(\omega')P(t, \omega, d\omega'), \qquad f \in L^2_\mu$$

form a semigroup for $t \geqslant 0$ having the following properties:

(i) W_t preserves positivity (i.e., $f(\omega) \geqslant 0$ implies $W_t f \geqslant 0$ for $t \geqslant 0$);

(ii) $W_t 1 = 1$.

The evolution of the distribution functions $\tilde{\rho}$ under the Markov process is described now by the adjoint semigroup W_t^* which also preserves positivity since W_t does: $\tilde{\rho} \rightarrow \tilde{\rho}_t = W_t^* \tilde{\rho}$. Since the measure μ is an invariant measure for the process (or equivalently the *microcanonical distribution function* 1 is the equilibrium state of the process) we also have the following:

(iii) $W_t^* 1 = 1$.

Every Markov process on Γ with stationary distribution μ is thus associated with a contraction semigroup satisfying the conditions (i)–(iii). Conversely every contraction semigroup W_t on L_μ^2 satisfying the above conditions comes from a stochastic Markov process, the transition probabilities $P(t, \omega, \Delta)$ being given by

$$P(t, \omega, \Delta) = (W_t \varphi_\Delta)(\omega)$$

Here φ_Δ denotes the characteristic (or indicator) function of the set Δ.

In the following we are interested in a special class of Markov processes whose semigroups W_t satisfy [in addition to the conditions (i)–(iii)] the following condition:

(iv) $\| W_t^*(\rho - 1) \|$ decreases *strictly* monotonically to 0 as $t \to \infty$ [$f(t)$ is *strictly* monotonically decreasing if $t < s \Rightarrow f(t) > f(s)$] for all states $\rho \not\equiv 1$ (i.e., for all nonnegative distribution functions $\rho \not\equiv 1$ with $\int_\Gamma \rho \, d\mu = 1$). This condition expresses the requirement that any initial distribution ρ tends strictly *monotonically* in time to the equilibrium distribution 1. For such processes the functional

$$\|\tilde{\rho}_t\|^2 = \int_\Gamma \tilde{\rho}_t^2 \, d\mu, \qquad (\tilde{\rho}_t = W_t^* \tilde{\rho})$$

and indeed any other convex functional including the usual expression for negative entropy

$$\int_\Gamma \tilde{\rho}_t \ln \tilde{\rho}_t \, d\mu$$

is an $\mathcal{H}$ function. Such Markov processes thus provide the best possible model of irreversible evolution obeying the law of monotonic increase of entropy.

Semigroups satisfying the conditions (i)–(iv) above have been called *strong* Markov semigroups in Ref. 1. The term *strong Markov process* is, however, generally used in probability theory in an entirely different sense. To avoid this confusion, we shall henceforward refer to semigroups satisfying conditions (i)–(iv) as *monotonic Markov semigroups*.

The intrinsic randomness of dynamical systems may now be defined in terms of the existence of a "change of representation" under which the unitary group associated with the deterministic dynamics is transformed into a monotonic Markov semigroup. More explicitly, the deterministic dynamics with induced unitary group U_t on L_μ^2 is said to be *intrinsically random* if there exists a (bounded) linear operator Λ defined on L_μ^2 such that

(a) Λ preserves positivity: $\Lambda\rho \geqslant 0$ if $\rho \geqslant 0$;

(b) $\int_\Gamma \rho \, d\mu = \int_\Gamma \Lambda\rho \, d\mu$;

(c) $\Lambda 1 = 1$;

(d) Λ has a densely defined inverse Λ^{-1};

(e) $\Lambda U_t \Lambda^{-1} = W_t^*$ (for $t \geqslant 0$) is a contraction semigroup satisfying the conditions (i)–(iv) formulated above.

The conditions (a) and (b) are necessary conditions for $\rho \to \Lambda\rho \equiv \tilde{\rho}$ to be interpreted as a change of representation. The condition (d) assures that the transformation does not involve "contraction of description" or coarse graining; the latter being given by projection operators. The condition (e) expresses the fact that under the change of representation $\rho \to \Lambda\rho$ the deterministic evolution $\rho \to U_t\rho$ is transformed into a stochastic Markovian evolution: $\Lambda\rho \to \Lambda U_t \rho = W_t^* \Lambda\rho$ described by W_t^*.

It may be noted that we have *not* required that Λ^{-1} should also preserve positivity. Indeed the following proposition shows that if one requires this then one is necessarily confined to the change of representations provided by (canonical) point transformations of the phase space onto itself which cannot, obviously, lead from deterministic dynamics to stochastic processes.

Proposition 1 [cf. Refs. 11 and 15]. Let Λ, in addition to satisfying conditions (a), (b), and (c) given above, also satisfy the following: (a′) Λ^{-1} preserves positivity, and (b′) the domain of Λ^{-1} contains all the characteristic functions φ_Δ ($\Delta \in \mathfrak{B}$). Then, there exists a measure-preserving point transformation T of Γ onto itself such that

$$(\Lambda\rho)(\omega) = \rho(T\omega)$$

Thus for intrinsically random systems the reverse passage $\tilde{\rho} \to \Lambda^{-1}\tilde{\rho}$, which leads from the stochastic description of the time evolution provided by W_t^* to the deterministic dynamics U_t, cannot physically be interpreted as a change of representation. One might say that the "reverse operation" leading to the deterministic description, though mathematically well defined, is a physically inadmissible operation. This is in conformity with the view that intrinsic randomness is a property only of highly unstable dynamical motion and in this situation the deterministic description of dynamic development constitutes an unphysical idealization. The above proposition shows also that $\Lambda U_t \Lambda^{-1} = W_t^*$, although defined for all t, can be required to be positivity preserving *only* either for positive t or for negative t but not for both at the same time.

Thus the passage to the stochastic description as contemplated in defining intrinsic randomness naturally breaks the symmetry between the "forward" and "backward" directions of time, causing the temporal evolution to be described by a semigroup rather than a group.

Let us now turn to the method of constructing the Λ transformation. A first remark to this end is that if $\Lambda U_t \Lambda^{-1}$ is a *monotonic* Markov semigroup then the operator

$$M \equiv \Lambda^* \Lambda$$

is necessarily a Lyapounov variable for U_t. In this connection, by a *Lyapounov variable* for the group U_t one means a positive operator M on L^2_μ such that the functional

$$\Omega(\rho_t) \equiv \langle U_t \rho, M U_t \rho \rangle$$

decreases *strictly* monotonically with t to its minimum value $\Omega(1)$ for all initial states $\rho \not\equiv 1$, the single exception being the microcanonical ensemble for which the functional Ω is, obviously, constant (cf. Ref. 10).

Intrinsic randomness thus implies the existence of Lyapounov variables which expresses the inherent irreversibility of the dynamical evolution. The existence of Lyapounov variables, on the other hand, is known[10] to imply that the system must be at least mixing in the sense of ergodic theory. In this way, one sees once again the close connection between intrinsic randomness and instability of motion: Instability leading to mixing on the phase space is a *necessary* condition for the manifestation of intrinsic randomness.

The foregoing remark shows that the Λ transformation (when it exists) may be constructed as essentially the "square root" of a Lyapounov variable. Now as described in Ref. 10, there is a general procedure for constructing the Lyapounov variables of a class of (abstract) dynamical systems, the so-called K flows. Briefly, for such systems one can construct an operator of "internal time" T which is defined to be a self-adjoint operator on $\mathcal{H}^{\perp}_{-\infty}$ satisfying the relation

$$U_t^* T U_t = T + tI \tag{2.1}$$

(Here $\mathcal{H}_{-\infty}$ denotes the one-dimensional subspace spanned by the constant functions and $\mathcal{H}^{\perp}_{-\infty}$ denotes the orthogonal complement of $\mathcal{H}_{-\infty}$.)

An operator M of the form

$$M = f(T) + P_{-\infty} \tag{2.2}$$

is then easily verified to be a Lyapounov variable if $f(T)$ is taken to be an operator function of T (in the sense of functional calculus) corresponding to a bounded positive function f decreasing strictly monotonically to 0, and $P_{-\infty}$ denotes the projection onto $\mathcal{H}_{-\infty}$.

The operator Λ defined to be the positive square root of an operator M

of the form (2.2) is, obviously, again of the same form:

$$\Lambda = h(T) + P_{-\infty} \tag{2.2'}$$

where $h = f^{1/2}$ is again a positive function decreasing strictly monotonically to 0. A possible strategy for establishing the intrinsic randomness of K flows would, therefore, be to show that for suitable choice of operator time T and of a monotonically decreasing positive function h the operator given by (2.2′) as well as the semigroup $W_t^* = \Lambda U_t \Lambda^{-1}$ (for $t \geqslant 0$) preserve positivity. We do this in the following section.

3. NONUNITARY EQUIVALENCE BETWEEN THE UNITARY GROUP INDUCED BY *K* FLOWS AND MARKOV SEMIGROUPS

Let $(\Gamma, \mathfrak{B}, \mu, T_t)$ be an abstract dynamical system. It is called a K flow (cf. Ref. 13) if there exists a σ-subalgebra $\mathfrak{F}_0 \subseteq \mathfrak{B}$ with the following properties:

(i) $T_t\mathfrak{F}_0 = \mathfrak{F}_t \subseteq T_s\mathfrak{F}_0 = \mathfrak{F}_s, \qquad t \leqslant s$

(ii) The σ-algebra generated by all $\mathfrak{F}_\lambda$ (with $-\infty < \lambda < +\infty$) coincides with the entire σ-algebra of measurable sets of the system:

$$\bigvee_{\lambda=-\infty}^{+\infty} \mathfrak{F}_\lambda = \mathfrak{B}$$

(iii) $\bigcap_{-\infty}^{\infty} \mathfrak{F}_\lambda = \mathfrak{F}_{-\infty}$ is the trivial σ-algebra which consists only of sets of measure 0 or complements of such sets.

Another characterization of K flows is that they have *completely positive* Kolmogorov entropy.[13] This latter characterization shows that the *observed* behavior of K flows contains an essential element of randomness. In fact, the complete positivity of Kolmogorov entropy may be interpreted to mean that the knowledge about the evolution of the system *in the past* obtained from an infinite repetition of *any realistic* measurement that corresponds to a partition of the phase space into a *finite* number of disjoint cells is insufficient for predicting the result one would obtain if the *same* measurement is performed in the future. Intuitively, then, the observed dynamical evolution of K systems is inherently nondeterministic in character. In the following we show that K flows are also intrinsically random in the stronger sense that the originally given deterministic evolution is equivalent—through a suitable change of representation which involves no coarse graining or contraction of description—to the stochastic evolution of a Markov process.

Denote by P_λ the orthogonal projection operator onto $L^2(\mathfrak{F}_\lambda, \mu)$, the subspace of functions in L^2_μ that are measurable with respect to $\mathfrak{F}_\lambda$. The properties (i)–(iii) of $\mathfrak{F}_\lambda$ then translate, respectively, into the following properties of P_λ:

(i′) $P_\lambda \leqslant P_{\lambda'}$ if $\lambda \leqslant \lambda'$;

(ii′) $\lim_{\lambda\to\infty} P_\lambda = I$;

(iii′) $\lim_{\lambda\to-\infty} P_\lambda = P_{-\infty}$ is the projection on the subspace spanned by the constant functions on Γ. Moreover, the obvious facts that

(iv) $T_t \mathcal{F}_\lambda = \mathcal{F}_{\lambda+t}$ and

(v) $T_t \mathcal{F}_{-\infty} = \mathcal{F}_{-\infty}$

imply, respectively, the following:

(iv′) $U_t^* P_\lambda U_t = P_{\lambda-t}$

(v′) $U_t^* P_{-\infty} U_t = P_{-\infty}$.

Now let F_λ stand for the projections $P_\lambda - P_{-\infty}$ (λ real). The family F_λ is then a spectral family of projections (or resolution of identity) in $\mathcal{H}_{-\infty}^{\perp}$. Moreover, owing to properties (iv′) and (v′), the spectral family F_λ is a *system of imprimitivity* for the group U_t:

$$U_t^* F_\lambda U_t = F_{\lambda-t} \tag{3.1}$$

The operator of "internal time" T mentioned in the preceding section may now be defined as the self-adjoint operator that has F_λ as its spectral family (cf. Ref. 10):

$$T = \int \lambda \, dF_\lambda \tag{3.2}$$

In fact, if T is given by (3.2) then the imprimitivity condition (3.1) on F_λ and the defining condition (2.1) on T are mutually equivalent.

As is well known, the functional calculus of the self-adjoint operator T associates with the function $h(\lambda)$ of a real variable an operator function $h(T)$ of T which is given by

$$h(T) = \int h(\lambda) \, dF_\lambda \tag{3.3}$$

To establish the intrinsic randomness of K flows let us now study the positivity-preserving property of operator Λ of the form

$$\Lambda = h(T) + P_{-\infty} \tag{2.2′}$$

as well as of the semigroup $\Lambda U_t \Lambda^{-1} = W_t^*$ (for $t \geqslant 0$).

Theorem 1. Let $h(\lambda)$ be a positive and monotonically decreasing function with $h(-\infty) \leqslant 1$. Then the operator Λ of the form (2.2′) is positivity preserving: $f \geqslant 0$ a.e. implies $\Lambda f \geqslant 0$ a.e.

To prove this we start with the following lemma.

Lemma 2. The projection operator P_λ onto $L^2(\mathcal{F}_\lambda, \mu)$ is positivity preserving.

Proof of Lemma 2. Let $f \geqslant 0$ a.e. Define now a measure μ_f on the σ algebra $\mathfrak{F}_\lambda$ by

$$\mu_f(\Delta) = \int_\Delta f\,d\mu$$

for $\Delta \in \mathfrak{F}_\lambda$. It is clear that μ_f is a nonnegative measure on $\mathfrak{F}_\lambda$, and it is absolutely continuous with respect to the measure μ restricted to $\mathfrak{F}_\lambda$ [i.e., $\mu(\Delta) = 0$ for $\Delta \in \mathfrak{F}_\lambda$ implies $\mu_f(\Delta) = 0$].

Thus by Radon–Nykodym theorem, there exists an essentially unique function, say, $\tilde{f}$, having the following properties:

(i) $\tilde{f}$ is measurable with respect to the σ-algebra $\mathfrak{F}_\lambda$;

(ii) $\int_\Delta \tilde{f}\,d\mu = \mu_f(\Delta)$, $\quad \Delta \in \mathfrak{F}_\lambda$. [The essential uniqueness of $\tilde{f}$ means that if $\tilde{f}'$ is another function satisfying (i) and (ii) then $\tilde{f} = \tilde{f}'$ a.e. with respect to μ.]

This function $\tilde{f}$ is called the Radon–Nykodym derivative of μ_f with respect to μ; since both μ and μ_f are nonnegative measures $\tilde{f} \geqslant 0$ a.e. [$\tilde{f}$ is, by definition, the *conditional expectation* $E(f\,|\,\mathfrak{F}_\lambda)$ of f given $\mathfrak{F}_\lambda$.]

Now let us consider the function $P_\lambda f$; since it belongs to $L^2(\mathfrak{F}_\lambda, \mu)$ it is obviously measurable with respect to $\mathfrak{F}_\lambda$. Moreover,

$$\int_\Delta (P_\lambda f)(\omega)\,d\mu = \langle P_\lambda f, \varphi_\Delta \rangle = \langle f, P_\lambda \varphi_\Delta \rangle$$

But if $\Delta \in \mathfrak{F}_\lambda$ then $P_\lambda \varphi_\Delta = \varphi_\Delta$ so that $\int_\Delta P_\lambda f\,d\mu = \langle f, \varphi_\Delta \rangle = \mu_f(\Delta)$ for all $\Delta \in \mathfrak{F}_\lambda$. Thus owing to the essential uniqueness of Radon–Nykodym derivative,

$$P_\lambda f = \tilde{f} \geqslant 0 \quad \text{a.e.} \quad \blacksquare$$

Remark. The preceding argument is the standard recipe in probability theory for defining conditional expectations $E(f\,|\,\mathfrak{F}_\lambda)$ and identifying $P_\lambda f$ with $E(f\,|\,\mathfrak{F}_\lambda)$.

Proof of Theorem 1. It will suffice to show that $\langle \rho, \Lambda\rho' \rangle \geqslant 0$ for every pair of nonnegative functions ρ, ρ' in L^2_μ. Now, owing to the assumed form of Λ, we obtain

$$\langle \rho, \Lambda\rho' \rangle = \int h(\lambda)\,d\langle \rho, F_\lambda \rho' \rangle + \langle \rho, P_{-\infty}\rho' \rangle$$

Partial integration of the first term on the right then gives

$$\langle \rho, \Lambda\rho' \rangle = h(\infty)\langle \rho, F_\infty \rho' \rangle - \int \langle \rho, F_\lambda \rho' \rangle\,dh(\lambda) + \langle \rho, P_{-\infty}\rho' \rangle$$

Use of the definition $F_\lambda = P_\lambda - P_{-\infty}$ and rearrangement of terms finally yields

$$\langle \rho, \Lambda\rho' \rangle = -\int \langle \rho, P_\lambda \rho' \rangle\,dh(\lambda) + h(\infty)\langle \rho, \rho' \rangle + \langle \rho, P_{-\infty}\rho' \rangle(1 - h(-\infty)) \tag{3.4}$$

For nonnegative ρ and ρ' all the terms in the right of (3.4) are individually nonnegative. This is so for the first term because $\langle \rho, P_\lambda \rho' \rangle \geqslant 0$ (owing to the positivity-preserving property of P_λ and the fact that $h(\lambda)$ is chosen to be monotonically decreasing). Nonnegativity of the second and third terms is obvious. ■

The preceding argument actually proves the slightly stronger result that if $h(\lambda)$ is *strictly* monotonically decreasing then Λ is "positivity improving," i.e., for any nonnegative function $\rho'(\neq 0)$ and *any* measurable set Δ with $\mu(\Delta) > 0$,

$$\langle \varphi_\Delta, \Lambda\rho' \rangle = \int_\Delta \Lambda\rho' \, d\mu > 0$$

In fact, it follows from (3.4) with φ_Δ replacing ρ that

$$\langle \varphi_\Delta, \Lambda\rho' \rangle \geqslant -\int \langle \varphi_\Delta, P_\lambda \rho' \rangle \, dh(\lambda)$$

Since $h(\lambda)$ is *strictly* monotonically decreasing and $\langle \varphi_\Delta, P_\lambda \rho' \rangle \geqslant 0$ the integral on the right must be strictly positive unless $\langle \varphi_\Delta, P_\lambda \rho' \rangle = 0$ for at least some λ in *every interval.*

This latter possibility is, however, ruled out since

$$\begin{aligned} \lim_{\lambda \to -\infty} \langle \varphi_\Delta, P_\lambda \rho' \rangle &= \langle \varphi_\Delta, P_{-\infty} \rho' \rangle \\ &= \mu(\Delta)\Big(\int \rho' \, d\mu\Big) > 0 \end{aligned}$$

This result shows that the action of Λ is highly "delocalizing" in the following sense: Even if the support of ρ' is confined to an arbitrarily small volume of the phase space Γ the support of the transformed function covers the entire phase space. Obviously such a transformation cannot be the multiplication operator by a function on the phase space, not can it be the induced operator from an underlying point transformation of the phase space. It is because of such delocalizing action of Λ that one may expect the change of representation $\rho \to \Lambda\rho$ to lead from deterministic to stochastic dynamics.

Let us note that the requirements that $\Lambda 1 = 1$ and (since Λ is self-adjoint) $\int \rho \, d\mu = \int \Lambda\rho \, d\mu$ are an easy consequence of the construction of Λ. The existence of a densely defined inverse Λ^{-1} can also be easily assured by choosing $h(\lambda)$ to be strictly positive a.e. Thus to prove the intrinsic randomness of K flows it remains only to show that for a suitable choice of the function $h(\lambda)$ the operators $W_t^* = \Lambda U_t \Lambda^{-1}$ (for $t \geqslant 0$) form a *monotonic Markov semigroup* (see Section 2 for definition). This part of this argument is essentially the same as the corresponding considerations in Ref. 1. But we supply the details for the sake of completeness.

Now, $\Lambda U_t \Lambda^{-1} \equiv W_t^*$ are positivity-preserving for $t \geqslant 0$ if and only if the operators $U_t^* \Lambda U_t \Lambda^{-1}$ (for $t \geqslant 0$) preserve positivity. Using the imprimitivity condition (3.1) on F_λ one sees that

$$U_t^*\Lambda U_t = \int h(\lambda)\, d(U_t^* F_\lambda U_t) + P_{-\infty}$$
$$= \int h(\lambda + t)\, dF_\lambda + P_{-\infty}$$

Since, on the other hand,

$$\Lambda^{-1} = \int \frac{1}{h(\lambda)}\, dF_\lambda + P_{-\infty}$$

the operators $U_t^*\Lambda U_t \Lambda^{-1}$ are of the form

$$U_t^*\Lambda U_t \Lambda^{-1} = \int \frac{h(\lambda + t)}{h(\lambda)}\, dF_\lambda + P_{-\infty}$$

According to Theorem 1, the operators $U_t^*\Lambda U_t \Lambda^{-1}$ (and hence W_t^*) preserve positivity for $t \geqslant 0$ if $h(\lambda)$ is so chosen that the functions

$$\tilde{h}_t(\lambda) = h(\lambda + t)/h(\lambda)$$

are monotonically decreasing functions of λ for all $t \geqslant 0$. Finally, the requirement that for all states $\rho \not\equiv 1$ $(\rho \geqslant 0, \int \rho\, d\mu = 1)$ $\|W_t^*(\rho - 1)\| \to 0$ *strictly monotonically* with t can be assured by requiring $h(\lambda)$ to decrease *strictly monotonically* to 0 as $\lambda \to \infty$.

In fact, denoting $\rho - 1$ by ρ', we find

$$\|W_t^*(\rho - 1)\|^2 = \langle \Lambda U_t \Lambda^{-1}\rho', \Lambda U_t \Lambda^{-1}\rho' \rangle$$
$$= \langle \Lambda^{-1}\rho', U_t^* \Lambda^2 U_t \Lambda^{-1}\rho' \rangle$$
$$= \int h^2(\lambda)\, d\langle \Lambda^{-1}\rho', U_t^* F_\lambda U_t \Lambda^{-1}\rho' \rangle + \langle \Lambda^{-1}\rho', P_{-\infty}\Lambda^{-1}\rho' \rangle$$
$$= \int h^2(\lambda + t)\, d\langle \Lambda^{-1}\rho', F_\lambda \Lambda^{-1}\rho' \rangle + \langle \Lambda^{-1}\rho', P_{-\infty}\Lambda^{-1}\rho' \rangle$$

Now, the term $\langle \Lambda^{-1}\rho', P_{-\infty}\Lambda^{-1}\rho' \rangle$ vanishes identically since $\int \rho\, d\mu = 1$ implies that $\rho - 1 = \rho'$ (and hence $\Lambda^{-1}\rho'$) belongs to $\mathcal{H}_{-\infty}^{\perp}$. The first term decreases strictly monotonically to 0 as $t \to +\infty$ because the integrand $h^2(\lambda + t)$ is chosen to decrease strictly monotonically to 0 as t increases to ∞.

Summarizing the proceeding considerations we obtain the following.

Theorem 3. Let the abstract dynamical system $\{\Gamma, \mathcal{B}, \mu, T_t\}$ be a K flow, U_t the unitary group induced by T_t. Further, let F_λ and $P_{-\infty}$ be the projections defined in the beginning of this section and $h(\lambda)$ a function satisfying the following conditions:

(i) $h(\lambda)$ is strictly monotonically decreasing with $\lim_{\lambda\to\infty} h(\lambda) = 0$ and $h(-\infty) \leqslant 1$.

(ii) $h(\lambda+s)/h(\lambda) \equiv \tilde{h}_s(\lambda)$ is a monotonically decreasing function of λ for every $s \geqslant 0$. Suppose Λ is defined by

$$\Lambda = \int_{-\infty}^{+\infty} h(\lambda)\, dF_\lambda + P_{-\infty}$$

Then Λ is positivity improving, has a densely defined inverse Λ^{-1}, and the operators $\Lambda U_t \Lambda^{-1} = W_t^*$ form a *monotonic Markov semigroup* for $t \geqslant 0$.

The intrinsic randomness of K flows is therefore established once the class of functions $h(\lambda)$ satisfying the conditions (i) and (ii) is shown to be nonempty.

Now condition (ii) means that $[\ln h(\lambda+s) - \ln h(\lambda)]$ is decreasing for $s \geqslant 0$. In other terms, condition (ii) means that $h(\lambda)$ is logarithmically concave, i.e., $h(\lambda)$ is of the form

$$h(\lambda) = e^{-\phi(\lambda)}$$

with $\phi(\lambda)$ convex. Thus conditions (i) and (ii) can be satisfied by choosing $\phi(\lambda)$ to be a convex, positive function which increases to $+\infty$ as $\lambda \to \infty$. For example, $h(\lambda) = e^{-(e^\lambda)}$ satisfies conditions (i) and (ii). Finally it may be remarked that Theorem 3 and its proof remain valid also, essentially as stated, for K systems (discrete time).

4. ISOMORPHISMS OF MARKOV PROCESSES ASSOCIATED WITH *K* FLOWS

We have seen in the preceding sections how stochastic Markov processes can be obtained from K flows through nonunitary similarity transformations Λ. We show now that this procedure associates nonisomorphic Markov processes with nonisomorphic K flows. In this connection, two Markov processes with transition probabilities $P_1(t,\omega,\Delta)$ and $P_2(t,\omega',\Delta')$ [on state spaces $(\Gamma_1, \mathfrak{B}_1, \mu_1)$ and $(\Gamma_2, \mathfrak{B}_2, \mu_2)$, respectively] are said to be isomorphic if there exists a one to one map S from all of Γ_1 (except possibly a set of measure 0) onto the entire space Γ_2 (modulo again a set of measure 0) such that (i) both S and S^{-1} are measurable and measure-preserving and (ii) $P_1(t,\omega,\Delta) = P_2(t, S\omega, S\Delta)$.

Let $V: L^2_{\mu_1} \to L^2_{\mu_2}$ be the unitary transformation induced by S:

$$(V\rho)(\omega) = \rho(S^{-1}\omega)$$

for $\rho \in L^2_{\mu_1}$, $\omega \in \Gamma_2$.

It is easy to verify that the condition (ii) on transition probabilities P_i of isomorphic processes is equivalent to the condition $VW_t^{(1)}V^* = W_t^{(2)}$ where $W_t^{(i)}$ $(i = 1,2)$ is the semigroup associated with the Markov process having transition probability $P_i(t,\cdot,\cdot)$ (cf. Section 2).

Let $U_t^{(1)}$ and $U_t^{(2)}$ be, respectively, the unitary groups induced from K flows $T_t^{(1)}$ and $T_t^{(2)}$. We suppose that the Markov processes $\Lambda_1 U_t^{(1)} \Lambda_1^{-1} = W_t^{(1)*}$ and $\Lambda_2 U_t^{(2)} \Lambda_2^{-1} = W_t^{(2)*}$ obtained from the two K flows in question through Λ transformations discussed before are isomorphic. We show now that if this holds then the two K flows in question must themselves be isomorphic. Now, as explained in the preceding paragraph, the assumed isomorphism between the Markov processes means that

$$VW_t^{(1)*}V^{-1} = W_t^{(2)*} \tag{4.1}$$

where V is the unitary operator induced by a measure-preserving transformation S from the phase space of the flow $T_t^{(1)}$ to that of $T_t^{(2)}$. Using the fact that Λ_i $(i = 1,2)$ are functions of "operator time" of the respective K flows we easily find that

$$\begin{aligned} W_t^{(i)*} = \Lambda_i U_t^{(i)} \Lambda_i^{-1} &= U_t^{(i)}\left[U_t^{(i)*} \Lambda_i U_t^{(i)} \Lambda_i^{-1} \right] \\ &\equiv U_t^{(i)} \tilde{\Lambda}_i \end{aligned}$$

Here $\tilde{\Lambda}_i$ may be easily computed to be

$$\tilde{\Lambda}_i = \int \frac{h_i(\lambda + t)}{h_i(\lambda)}\, dF_\lambda^{(i)} + P_{-\infty}$$

with $F_\lambda^{(i)}$ denoting the spectral family of the "operator time" of K flow $U_t^{(i)}$ and

$$\Lambda_i = \int h_i(\lambda)\, dF_\lambda^{(i)} + P_{-\infty}$$

Thus $\tilde{\Lambda}_i$ is a positive and self-adjoint operator and $W_t^{(i)*} = U_t^{(i)} \tilde{\Lambda}_i$ with $U_t^{(i)}$ unitary and $\tilde{\Lambda}_i \geqslant 0$ represents the "*polar decomposition*" of the operator $W_t^{(i)*}$. The important point is that the unitary part in the polar decomposition of $W_t^{(i)*}$ coincides with the group $U_t^{(i)}$, this result being a consequence of the fact that Λ_i are functions of operator time. Condition (4.1) may be rewritten now as

$$\begin{aligned} \left(VU_t^{(1)}V^* \right)\left(V\tilde{\Lambda}_1 V^* \right) &= W_t^{(2)*} \\ &= U_t^{(2)} \tilde{\Lambda}_2 \end{aligned}$$

Since $VU_t^{(1)}V^*$ is unitary and $V\tilde{\Lambda}_1 V^* \geqslant 0$ both sides of the above equation provide a polar decomposition of $W_t^{(2)*}$. Uniqueness of polar decomposition then entails

$$VU_t^{(1)}V^* = U_t^{(2)}$$

which means that the transformation S from which V is induced satisfies

$$ST_t^{(1)}S^{-1} = T_t^{(2)}$$

In other terms, the two K flows are isomorphic. We have thus established the following proposition.

Proposition 1.[3] Markov processes obtained, through nonunitary Λ transformation described in Section 3, from *nonisomorphic* K flows, are necessarily nonisomorphic.

The preceding discussion shows also that the symmetry group of the Markov process $W_t^* = \Lambda U_t \Lambda^{-1}$ associated with the K flow T_t is necessarily a *subgroup* of the symmetry group of the K flow in question.

In fact, a symmetry operation of the Markov process would correspond to a (positivity-preserving) unitary operator V such that $VW_t^* V^{-1} = W_t^*$. But this implies, as before, that $VU_tV^{-1} = U_t$, which means that V is a symmetry operation for the K flow too. One thus finds that the passage to the stochastic description through nonunitarity transformation as described in Ref. 1 and here is expected to lead also to symmetry breaking. We have already remarked that this passage necessarily involves breaking the symmetry between positive and negative directions of time. It would be interesting to examine whether additional symmetries, such as space inversion symmetry, can be broken by this procedure.[(16)]

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[3] In the special case of Bernoulli systems this proposition has been previously obtained by explicity calculation; see Ref. 14.

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